Mutually unbiased bases and discrete Wigner functions

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(Dated: February 1, 2008)

Mutually unbiased bases and discrete Wigner functions are closely, but not uniquely related. Such a connection becomes more interesting when the Hilbert space has a dimension that is a power of a prime $N = d^n$, which describes a composite system of n qudits. Hence, entanglement naturally enters the picture. Although our results are general, we concentrate on the simplest nontrivial example of dimension $N = 8 = 2^3$. It is shown that the number of fundamentally different Wigner functions is severely limited if one simultaneously imposes translational covariance and that the generating operators consist of rotations around two orthogonal axes, acting on the individual qubits only.

PACS numbers:

I. INTRODUCTION

Quasiprobability distributions have been useful in quantum mechanics both as visualization tools and for computational purposes. Typical examples include the Glauber-Sudarshan [1, 2], the Husimi [3], and the Wigner function [4]. These are all functions of continuous parameters that map quantum states onto a continuous phase space.

The growing interest in quantum information has fueled the search for discrete phase-space counterparts of these distributions [5]. In particular, the discrete Wigner function has attracted a good deal of attention. There is no unique way of defining such a function: one family of methods maps the states of the Hilbert space onto a phase-space grid of $2N \times 2N$ points (implying a redundancy of information) [6, 7, 8], whereas it is also possible to map the states onto a (nonredundant) $N \times N$ phase space [9, 10, 11, 12, 13, 14]. In this paper we shall investigate the latter type of Wigner functions.

Even-dimensional Hilbert spaces are of special interest to visualize the effects of quantum information processing. The ubiquitous qubit lives in a two-dimensional Hilbert space, and n qubits hence span a 2^n-dimensional space. Wigner functions for two and three qubits have found applications in providing solutions to, e. g., state reconstruction [15], quantum teleportation [16, 17], quantum optics [18], quantum computing [19], and the mean king problem [20, 21, 22, 23, 24, 25].

In several of these contexts, mutually unbiased bases (MUBs) are also of interest [26, 27]. We recall that each vector in one of these bases is an equal-magnitude superposition of all the vectors in any of the other bases. MUBs are central to quantum tomography and state reconstruction [28], but are also valuable for quantum key distribution [29, 30, 31, 32]. It is known that when the Hilbert space dimension N is a prime, or a power of a prime, there exist exactly $N + 1$ MUBs [33]. In this paper we shall mainly discuss the three-qubit case, for it is the smallest space consisting of qubits where different MUB structures, with respect to their entanglement properties, exist [34, 35]. More precisely, in this space there are four different MUB structures corresponding to $s = 0, 1, 2$, and 3, respectively, where s is the number of triseparable MUBs. The fact that different structures are possible has strong implications for the physical implementation of quantum tomographic measurements and quantum information protocols. It also affects the manner in which we can map states onto a phase space while retaining some properties that we consider "natural" in the continuous case.

The rest of the paper is organized as follows: In Section II the continuous Wigner function and specifically the concept of translational covariance is recalled. In Section III a method for determining MUBs for three qubits is briefly reviewed. The concept involves operator sets based on primitive, one-qubit operations. In Sec. IV we devise a construction of the Wigner function by associating the primitive operators (that generate a certain MUB) with translations in phase space. We show that this MUB has a unique structure, once we disregard the somewhat trivial degrees of freedom related to qubit labeling and to association of lines or curves in phase space to states. In Section V we derive two different Wigner functions from a different MUB structure, while in Section VI we discuss the remaining two MUB structures, having $s = 0$ or $s = 1$ triseparable bases. In this case, the principle of translational invariance no longer works in association with translations corresponding to single qubit MUB-generating operators. Finally, we make some concluding remarks in Section VII.

II. THE CONTINUOUS WIGNER FUNCTIONS

The continuous Wigner function has three properties that we are particularly keen on retaining when constructing a discrete counterpart. The function is real, when integrated along any direction in phase space it yields a nonnegative function with unit area, i.e., a marginal probability distribution, and finally it is translationally covariant. The latter property can be stated in mathematical terms as follows. Let $W(q, p)$ be the Wigner function (expressed in terms of the position q and the momentum p) corresponding to a density matrix $\hat\rho$, and let $\hat\rho'$ obtained from ρ by a displacement (q_0, p_0) in phase space:

$$\hat\rho' = \exp[i(q_0\hat{p} - p_0\hat{q})/\hbar]\,\hat\rho\,\exp[-i(q_0\hat{p} - p_0\hat{q})/\hbar], \quad (2.1)$$

where $\hat{q}$ and $\hat{p}$ are the position and momentum operators, respectively. Then the Wigner function W' corresponding to $\hat\rho'$ is obtained from W via the transformation

$$W'(q, p) = W(q - q_0, p - p_0). \quad (2.2)$$

In other words, when the density matrix is translated, the Wigner function follows along rigidly.

III. MUB STRUCTURES FOR THREE QUBITS

Mutually unbiased bases can be constructed using a number of methods that depend on the dimensionality of the space. The main dividing lines are if the dimension is prime, a product of primes, or a power of a prime, and, in the latter case, if it is odd or even [36, 37, 38, 39, 40, 41, 42, 43, 44, 45]. The problem appears to be closely related to mutually orthogonal Latin squares [46, 47] and to the existence of finite projective planes of certain orders [48, 49]. As stated in the Introduction, we confine our study to the case of three qubits, that is, to an eight-dimensional Hilbert space. In this Hilbert space there exist four MUB structures, where the word "structure" denotes sets of MUBs where the basis vectors are either triseparable (i.e., factorizable in a tensor product of three individual qubit states), separable in one qubit and one maximally entangled two-qubit state (i.e., biseparable), or nonseparable [34, 35]. The four structures are (3,0,6), (2,3,4), (1,6,2), and (0,9,0), where the labels indicate the number of triseparable, biseparable, and nonseparable bases, respectively. Note that all of them are of the form $(s, b, 9 - s - b)$, where b is the number of biseparable bases. We shall primarily concentrate on the two first structures, since they correspond to MUBs where at least two bases are fully separable so we can associate theses bases with local properties of the three qubits. This simplifies the analogy with the continuous Wigner function.

We follow the construction algorithm worked out by Klimov, Sánchez-Soto, and de Guise in Ref. 42 to generate, in a systematic way, a set of operators whose eigenvectors constitute a MUB. If we take a spin-1/2 system

TABLE I: Nine sets of operators defining a (2,3,4) MUB.

$\hat\sigma_z\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\mathbb{1}}\hat\sigma_z$	$\hat{\mathbb{1}}\hat\sigma_z\hat{\mathbb{1}}$	$\hat\sigma_z\hat{\mathbb{1}}\hat\sigma_z$	$\hat{\mathbb{1}}\hat\sigma_z\hat\sigma_z$	$\hat\sigma_z\hat\sigma_z\hat\sigma_z$	$\hat\sigma_z\hat\sigma_z\hat{\mathbb{1}}$
$\hat\sigma_x\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat\sigma_x\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\mathbb{1}}\hat\sigma_x$	$\hat\sigma_x\hat\sigma_x\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat\sigma_x\hat\sigma_x$	$\hat\sigma_x\hat\sigma_x\hat\sigma_x$	$\hat\sigma_x\hat{\mathbb{1}}\hat\sigma_x$
$\hat\sigma_y\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat\sigma_x\hat\sigma_z$	$\hat{\mathbb{1}}\hat\sigma_z\hat\sigma_x$	$\hat\sigma_y\hat\sigma_x\hat\sigma_z$	$\hat{\mathbb{1}}\hat\sigma_y\hat\sigma_y$	$\hat\sigma_y\hat\sigma_y\hat\sigma_y$	$\hat\sigma_y\hat\sigma_z\hat\sigma_x$
$\hat\sigma_x\hat{\mathbb{1}}\hat\sigma_z$	$\hat{\mathbb{1}}\hat\sigma_y\hat{\mathbb{1}}$	$\hat\sigma_z\hat{\mathbb{1}}\hat\sigma_y$	$\hat\sigma_x\hat\sigma_y\hat\sigma_z$	$\hat\sigma_z\hat\sigma_y\hat\sigma_y$	$\hat\sigma_y\hat\sigma_y\hat\sigma_x$	$\hat\sigma_y\hat{\mathbb{1}}\hat\sigma_x$
$\hat\sigma_x\hat\sigma_z\hat{\mathbb{1}}$	$\hat\sigma_z\hat\sigma_x\hat\sigma_z$	$\hat{\mathbb{1}}\hat\sigma_z\hat\sigma_y$	$\hat\sigma_y\hat\sigma_y\hat\sigma_z$	$\hat\sigma_z\hat\sigma_y\hat\sigma_x$	$\hat\sigma_y\hat\sigma_x\hat\sigma_x$	$\hat\sigma_x\hat{\mathbb{1}}\hat\sigma_y$
$\hat\sigma_y\hat{\mathbb{1}}\hat\sigma_z$	$\hat{\mathbb{1}}\hat\sigma_y\hat\sigma_z$	$\hat\sigma_z\hat\sigma_z\hat\sigma_y$	$\hat\sigma_y\hat\sigma_y\hat{\mathbb{1}}$	$\hat\sigma_z\hat\sigma_x\hat\sigma_x$	$\hat\sigma_x\hat\sigma_x\hat\sigma_y$	$\hat\sigma_x\hat\sigma_z\hat\sigma_x$
$\hat\sigma_x\hat\sigma_z\hat\sigma_z$	$\hat\sigma_z\hat\sigma_y\hat\sigma_z$	$\hat\sigma_z\hat\sigma_z\hat\sigma_x$	$\hat\sigma_y\hat\sigma_x\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat\sigma_x\hat\sigma_y$	$\hat\sigma_x\hat\sigma_y\hat\sigma_x$	$\hat\sigma_y\hat{\mathbb{1}}\hat\sigma_y$
$\hat\sigma_y\hat\sigma_z\hat\sigma_z$	$\hat\sigma_z\hat\sigma_y\hat{\mathbb{1}}$	$\hat\sigma_z\hat{\mathbb{1}}\hat\sigma_x$	$\hat\sigma_x\hat\sigma_x\hat\sigma_z$	$\hat{\mathbb{1}}\hat\sigma_y\hat\sigma_x$	$\hat\sigma_y\hat\sigma_x\hat\sigma_y$	$\hat\sigma_x\hat\sigma_z\hat\sigma_y$
$\hat\sigma_y\hat\sigma_z\hat{\mathbb{1}}$	$\hat\sigma_z\hat\sigma_x\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\mathbb{1}}\hat\sigma_y$	$\hat\sigma_x\hat\sigma_y\hat{\mathbb{1}}$	$\hat\sigma_z\hat\sigma_x\hat\sigma_y$	$\hat\sigma_x\hat\sigma_y\hat\sigma_y$	$\hat\sigma_y\hat\sigma_z\hat\sigma_y$

as our model for a qubit (described, apart from a factor $\hbar/2$, by the Pauli operators $\hat{\boldsymbol{\sigma}}$), for three qubits it suffices to consider tensor products of $\hat{\mathbb{1}}^{(i)}, \hat\sigma_x^{(i)}, \hat\sigma_y^{(i)}, \hat\sigma_z^{(i)}$, where the superscript $i = 1, 2, 3$ labels the qubit, in such a way that the tensor product contains one operator acting on each qubit. For example, $\hat\sigma_x^{(1)} \otimes \hat{\mathbb{1}}^{(2)} \otimes \hat\sigma_y^{(3)}$ is a legitimate operator in this respect, which we shall write from now as $\hat\sigma_x\hat{\mathbb{1}}\hat\sigma_y$.

We wish now to construct MUBs where two of the bases are generated by individual qubit rotations around two orthogonal axes, which we arbitrarily take as the x and the z axes. Hence, we define the MUB using the operators $\{\hat\sigma_z\hat{\mathbb{1}}\hat{\mathbb{1}}, \hat{\mathbb{1}}\hat\sigma_z\hat{\mathbb{1}}, \hat{\mathbb{1}}\hat{\mathbb{1}}\hat\sigma_z\}$ and $\{\hat\sigma_x\hat{\mathbb{1}}\hat{\mathbb{1}}, \hat{\mathbb{1}}\hat\sigma_x\hat{\mathbb{1}}, \hat{\mathbb{1}}\hat{\mathbb{1}}\hat\sigma_x\}$. It can be shown that fundamentally, there is only a single way of accomplishing this, namely from the construction in Table I.

The algorithm guarantees that the simultaneous eigenstates of the operators in each of the nine rows give a complete basis, and each basis is mutually unbiased to each other. From earlier work [34, 35], we know that with respect to separability, Table I defines a (2,3,4) structure.

Moreover, the table is uniquely defined by the 2×3 entries in the three first columns of the first two rows. Indeed this is the case, since the operators in the first two rows are related by $\hat{O}_{r,c} = \hat{O}_{r,c-2}\hat{O}_{r,c-3}$, where $r = 1, 2$ and $c = 1, \ldots, 7$ denote the row and column of the operator, respectively, and must be taken modulo seven. Subsequently, the other rows can be expressed as $\hat{O}_{r,c} = \hat{O}_{2,c}\hat{O}_{1,c+r-3}$ for $2 < r \le 9$ (where all phase factors arising from the products have been neglected). In what follows, we write explicitly only the first 2×3 elements of each basis to save space.

We observe that if the first and second row are interchanged, the new table defines the same structure. Our choice of axes is of course arbitrary, we could just as well denote the z axis by x and vice versa. If we permute the labeling of the axes, the result is the first set in Table II. This looks different from Table I, and for fixed x and z axes it generates a different MUB than Table I. However, at the fundamental level, both are the same structure. If a certain election of the axes orientation is made, the MUBs are identical.

Another arbitrary choice we have made is the labeling

TABLE II: The five different sets resulting from a rearranging of the three qubits of Table I as 132, 213, 312, 321, and 231, respectively. Only the relevant 2×3 operators are shown.

$\hat{\sigma}_z\hat{\mathbb{1}}\hat{\mathbb{1}}$ $\hat{\mathbb{1}}\hat{\sigma}_z\hat{\mathbb{1}}$ $\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_z$	$\hat{\mathbb{1}}\hat{\sigma}_z\hat{\mathbb{1}}$ $\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_z$ $\hat{\sigma}_z\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\sigma}_z\hat{\mathbb{1}}$ $\hat{\sigma}_z\hat{\mathbb{1}}\hat{\mathbb{1}}$ $\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_z$	$\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_z$ $\hat{\sigma}_z\hat{\mathbb{1}}\hat{\mathbb{1}}$ $\hat{\mathbb{1}}\hat{\sigma}_z\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_z$ $\hat{\mathbb{1}}\hat{\sigma}_z\hat{\mathbb{1}}$ $\hat{\sigma}_z\hat{\mathbb{1}}\hat{\mathbb{1}}$
$\hat{\sigma}_x\hat{\mathbb{1}}\hat{\mathbb{1}}$ $\hat{\mathbb{1}}\hat{\sigma}_x\hat{\mathbb{1}}$ $\hat{\mathbb{1}}\hat{\sigma}_x\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\sigma}_x\hat{\mathbb{1}}$ $\hat{\sigma}_x\hat{\mathbb{1}}\hat{\mathbb{1}}$ $\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_x$	$\hat{\mathbb{1}}\hat{\sigma}_x\hat{\mathbb{1}}$ $\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_x$ $\hat{\sigma}_x\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_x$ $\hat{\mathbb{1}}\hat{\sigma}_x\hat{\mathbb{1}}$ $\hat{\sigma}_x\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_x$ $\hat{\sigma}_x\hat{\mathbb{1}}\hat{\mathbb{1}}$ $\hat{\mathbb{1}}\hat{\sigma}_x\hat{\mathbb{1}}$

of the qubits. If the qubits are numbered 1, 2, and 3, from left to right in Table I, rearranging the qubits (and the corresponding operators) in the orders 132, 213, 312, 321, and 231 will then result in the five sets shown in Table II. We see that, due to the simple structure of the tables, a permutation of the axes can also be seen as a relabeling of the qubits.

The sets in Tables I and II are the only three-qubit, MUB generating matrices containing in one row the single qubit rotations around one axis, e.g., $\{\hat{\sigma}_z\hat{\mathbb{1}}\hat{\mathbb{1}}, \hat{\mathbb{1}}\hat{\sigma}_z\hat{\mathbb{1}}, \hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_z\}$ (in any order), and in the next row the single qubit rotations around an orthogonal axis, e.g., $\{\hat{\sigma}_x\hat{\mathbb{1}}\hat{\mathbb{1}}, \hat{\mathbb{1}}\hat{\sigma}_x\hat{\mathbb{1}}, \hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_x\}$. Hence, with this restriction, only one MUB structure is allowed (up to a relabeling of the qubits). This is our first important result.

IV. A THREE-QUBIT WIGNER FUNCTION

When the Hilbert space is finite, there are several possibilities for defining a Wigner function, as discussed in the Introduction. We follow the approach of Wootters, Gibbons, and Hoffman [13, 14] and try to incorporate the three features discussed for the continuous case. To generate Wigner functions from the MUB in Table I, we resort to elementary notions of finite fields. The field $\mathbb{F}_8$ with exactly eight elements can be seen as the set $\{0, 1, \mu, \dots, \mu^6\}$ (which is also an additive group), where the primitive element μ is a root of the following irreducible polynomial on $\mathbb{Z}_2$ (the integers modulo 2)

$$\theta^3 + \theta + 1 = 0. \tag{4.1}$$

With this arithmetic, we have

$$1 = \mu^3 + \mu^5 + \mu^6, \qquad \mu = \mu^5 + \mu^6,$$
$$\mu^2 = \mu^3 + \mu^5, \qquad \mu^4 = \mu^3 + \mu^6. \tag{4.2}$$

Note that the subset $\{\mu^3, \mu^5, \mu^6\}$ defines a self-dual basis:

$$\operatorname{tr}(\mu^i \mu^j) = \delta_{ij}, \qquad i, j \in 3, 5, 6. \tag{4.3}$$

Here the trace of an element θ in this field is defined as

$$\operatorname{tr}\theta = \theta + \theta^2 + \theta^4. \tag{4.4}$$

Following Refs. 13 and 14, we subsequently associate the self-dual basis elements, in a two-dimensional phase-space coordinate representation, with the two groups of three operators, each one representing an "axis" in phase space:

$$(\mu^3, 0) \leftrightarrow \hat{\sigma}_x\hat{\mathbb{1}}\hat{\mathbb{1}},$$
$$(\mu^5, 0) \leftrightarrow \hat{\mathbb{1}}\hat{\sigma}_x\hat{\mathbb{1}}, \tag{4.5}$$
$$(\mu^6, 0) \leftrightarrow \hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_x,$$

and

$$(0, \mu^3) \leftrightarrow \hat{\sigma}_z\hat{\mathbb{1}}\hat{\mathbb{1}},$$
$$(0, \mu^5) \leftrightarrow \hat{\mathbb{1}}\hat{\sigma}_z\hat{\mathbb{1}}, \tag{4.6}$$
$$(0, \mu^6) \leftrightarrow \hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_z.$$

We map the eight-dimensional Hilbert space onto a 8×8 discrete phase space, and label the x (horizontal) and z axes (vertical) with the field element sequence $\{0, 1, \mu, \mu^2, \dots, \mu^6\}$. This gives us a phase-space coordinate in terms of field-element powers: each phase-space point has a unique coordinate, such as (μ^3, μ^6), as can be seen in Fig. 1.

One can now construct lines and striations (sets of eight parallel lines, i.e., with no point in common) based on the algorithm for the MUBs. The striations are uniquely defined by (any) one line and the requirement of translational covariance. We shall, for definiteness, take the striation-defining line as the one that includes the origin, that is, a ray. Subsequently, the seven other lines belonging to each striation can be generated by translating the ray in a manner that will be described below. In order to find a correspondence between the phase-space lines, the translations, and MUBs, we first define the unitary qubit-flip operators corresponding to the two sets of generating operators (that is, the set of operators in a row in the respective defining table). The unitary operator $\exp(i\pi\hat{\sigma}_x/2) \otimes \hat{\mathbb{1}} \otimes \hat{\mathbb{1}} = \hat{U}_x \otimes \hat{\mathbb{1}} \otimes \hat{\mathbb{1}}$ (with $\hat{U}_x = i\hat{\sigma}_x$) rotates the first spin qubit around the x axis

μ^6	2	6	7	8	4	5	3	9
μ^5	2	7	8	9	5	6	4	3
μ^4	2	8	9	3	6	7	5	4
μ^3	2	5	6	7	3	4	9	8
μ^2	2	4	5	6	9	3	8	7
μ	2	9	3	4	7	8	6	5
1	2	3	4	5	8	9	7	6
0	o	1	1	1	1	1	1	1
	0	1	μ	μ^2	μ^3	μ^4	μ^5	μ^6

FIG. 1: The striation-generating curves corresponding to the MUB construction defined by Table 1.

by the angle π. That is, the operator is an eigenoperator of the set $\{\hat{\sigma}_x\hat{\mathbb{1}}\hat{\mathbb{1}}, \hat{\mathbb{1}}\hat{\sigma}_x\hat{\mathbb{1}}, \hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_x\}$, and flips the first qubit of any eigenstate of $\{\hat{\sigma}_z\hat{\mathbb{1}}\hat{\mathbb{1}}, \hat{\mathbb{1}}\hat{\sigma}_z\hat{\mathbb{1}}, \hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_z\}$ into the orthogonal qubit state. In the same manner, local unitary flip operators corresponding to the remaining five basis-defining operators $\{\hat{\mathbb{1}}\hat{\sigma}_x\hat{\mathbb{1}}, \hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_x, \hat{\sigma}_z\hat{\mathbb{1}}\hat{\mathbb{1}}, \hat{\mathbb{1}}\hat{\sigma}_z\hat{\mathbb{1}}, \hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_z\}$ can be constructed.

In the phase-space representation these flip operators are represented by a translation by the corresponding field element "vectors", e.g., $\hat{U}_x \otimes \hat{\mathbb{1}} \otimes \hat{\mathbb{1}} \leftrightarrow$ translation by addition of $(\mu^3, 0)$, and $\hat{\mathbb{1}} \otimes \hat{U}_z \otimes \hat{\mathbb{1}} \leftrightarrow$ translation by addition of $(0, \mu^5)$. The ray associated with the set $\{\hat{\sigma}_x\hat{\mathbb{1}}\hat{\mathbb{1}}, \hat{\mathbb{1}}\hat{\sigma}_x\hat{\mathbb{1}}, \hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_x\}$ is hence generated by translation of the origin $\{0,0\}$ by all additive (modulo 2) combinations of $(\mu^3, 0)$, $(\mu^5, 0)$, and $(\mu^6, 0)$. It is clear that this will result in the horizontal line (ray) of Fig. 1. (The origin is marked with an o because it belongs to all striation-generating rays.) It is also evident that if every point on this ray is transformed by any combination of the three translations, the ray will remain invariant. This reflects the fact that the state associated with the ray is an eigenvector of the corresponding unitary qubit flip operators.

The rest of the lines of this striation can be obtained by translation by all the combinations of the three vectors $(0, \mu^3)$, $(0, \mu^5)$, and $(0, \mu^6)$ (respecting the modulo 2 arithmetic and the field algebra). This results in a set of eight parallel lines – the striation corresponding to the basis generated by the operators $\{\hat{\sigma}_x\hat{\mathbb{1}}\hat{\mathbb{1}}, \hat{\mathbb{1}}\hat{\sigma}_x\hat{\mathbb{1}}, \hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_x\}$. In the same manner, one can construct the striation (consisting of straight vertical lines) representing the MUB generated by $\{\hat{\sigma}_z\hat{\mathbb{1}}\hat{\mathbb{1}}, \hat{\mathbb{1}}\hat{\sigma}_z\hat{\mathbb{1}}, \hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_z\}$. The ray belonging to this set is labeled with 2s in Fig. 1.

To generate the third striation-generating "ray", we follow the recipe given above [42] and simply multiply the first two rows of Table I columnwise. In doing so, we ignore overall phase factors. We get the third set of MUB-generating operators $\{\hat{\sigma}_y\hat{\mathbb{1}}\hat{\mathbb{1}}, \hat{\mathbb{1}}\hat{\sigma}_x\hat{\sigma}_z, \hat{\mathbb{1}}\hat{\sigma}_z\hat{\sigma}_x\}$. These operators correspond to the phase-space operators (μ^3, μ^3), (μ^5, μ^6), and (μ^6, μ^5). The "ray" generated by displacement of the origin by all the combinations of these translations is labeled with 3s in Fig. 1.

The reason we have used the word "ray" within quotation marks in the paragraph above is that "ray" 3 is in fact a curve in phase-space. If a point belonging to it is denoted (α, β), then the curve is parametrically defined as

$$\alpha = \mu^3\kappa + \mu^5\kappa^2 + \mu^6\kappa^4, \qquad \beta = \mu^2\kappa + \kappa^2 + \mu^4\kappa^4, \quad (4.7)$$

where κ is a parameter running over the field elements. This parametrization corresponds to the explicit expression

$$\beta^2 + \mu\beta = \alpha^2 + \mu\alpha, \qquad (4.8)$$

or, equivalently,

$$\beta = \mu^6\alpha + \mu^5\alpha^4 + \mu^3\alpha^2, \qquad (4.9)$$

from which it is clear that α and β are not linearly related. In the following we will refer to a curve passing through the origin as a homogeneous curve.

From the table we see that, as required by a set of MUBs, each curve crosses the curves of the other striations only once, at the origin in Fig. 1 by our choice of depicting only homogeneous curves. Every full striation can be generated from the homogeneous curve by translation in the horizontal direction by adding, in sequence, all the combinations of $(\mu^3, 0)$, $(\mu^5, 0)$, and $(\mu^6, 0)$. In physical space, this corresponds to all possible combinations of spin flips around the x axes. The striations can equally well be generated by vertical translation by all the combinations of $(0, \mu^3)$, $(0, \mu^5)$, and $(0, \mu^6)$, corresponding to spin flips around the z axes. In fact, we can generate any striation from the corresponding homogeneous curve by the translations corresponding to any of the other curve-generating operators. For example, the operators $(\mu^3, \mu^3 + \mu^5)$, (μ^5, μ^3), and (μ^6, μ^6), corresponding to the first three columns of the bottom row of Table I will also generate the same striation. This follows from the fact that the relation between the bases is the same for any two of them. The fully separable bases are not particular in this respect.

To obtain a Wigner function from the striations one more step is needed, namely, to associate each line in phase space with a state. Again this involves an arbitrary choice that does not lead to anything fundamentally new. As described in Ref. 14, we can associate any basis eigenstate to any line in a striation, but once this choice is made, the state associated to a line obtained by a particular translation must correspond to the state obtained from the first by the corresponding spin-flip transformation. For example, if we associate the vertical ray in Fig. 1 (labeled with 2s) with the state $|\uparrow\uparrow\uparrow\rangle_z$ (in a spin-1/2 representation with z as our spin axis), then the vertical line that includes the point $(\mu^3 + \mu^6, 0) = (\mu^4, 0) \leftrightarrow \hat{U}_x \otimes \hat{\mathbb{1}} \otimes \hat{U}_x$ represents the state $\hat{U}_x \otimes \hat{\mathbb{1}} \otimes \hat{U}_x|\uparrow\uparrow\uparrow\rangle_z = |\downarrow\uparrow\downarrow\rangle_z$. As there are $N+1$ striations, and for each striation we can associate the generating ray with any of N state vectors, there are N^{N+1} *quantum nets* associated with this choice of phase-space bases (single qubit rotations around the x and z axes). Of these, one can group the possibilities in N^{N-1} *equivalence classes* each containing N^2 elements [14]. The elements are simply the $N \times N$ possible discrete translations of the space using all combinations of the horizontal and vertical translation operators. Nevertheless, we stress that in our approach, in contrast to that of Wootters [13, 14], we associate curves in phase space (and not merely lines) to states.

With the prerequisites above, for any phase-space point (α, β) we can define Hermitian phase-space point-operators $\hat{A}_{(\alpha, \beta)}$ as

$$\hat{A}_{(\alpha, \beta)} = \sum_{k=1}^{9} \hat{\rho}_{k, (\alpha, \beta)} - \hat{\mathbb{1}}, \qquad (4.10)$$

where $\hat{\rho}_{k,(\alpha,\beta)}$ is the density matrix associated with the curve belonging to striation k passing through (α,β). From the mutual unbiasedness of the orthonormal bases, it follows that

$$\mathrm{Tr}[\hat{A}_{(\alpha,\beta)}] = 1,$$

$$\mathrm{Tr}[\hat{A}_{(\alpha,\beta)}\hat{A}_{(\alpha',\beta')}] = 8\delta_{\alpha,\alpha'}\delta_{\beta,\beta'}, \quad (4.11)$$

where Tr (with upper case) denotes the ordinary Hilbert-space trace operation. The discrete Wigner function $W(\alpha,\beta)$ of the density matrix $\hat{\rho}$ is then defined as

$$W(\alpha,\beta) = \frac{1}{8}\mathrm{Tr}[\hat{\rho}\hat{A}_{(\alpha,\beta)}]. \quad (4.12)$$

This leads to the relations

$$\sum_{\alpha,\beta} W(\alpha,\beta) = 1,$$
$$\hat{\rho} = \sum_{\alpha,\beta} W(\alpha,\beta)\hat{A}_{(\alpha,\beta)}, \quad (4.13)$$
$$\mathrm{Tr}(\hat{\rho}\hat{\rho}') = 8\sum_{\alpha,\beta} W(\alpha,\beta)W'(\alpha,\beta),$$

where $W'(\alpha,\beta)$ is the Wigner function of the (arbitrary) density matrix $\hat{\rho}'$. It is seen that the phase-space point-operators, and consequently the Wigner function, follow naturally from the construction algorithm outlined above, based only on the generating operator tables, the irreducible polynomial (4.1), and the (arbitrary) association between basis states and phase-space rays.

The phase-space structure corresponding to the first set in Table II is obtained by making a mirror image of Fig. 1 with respect to the diagonal through the origin. As noted above, this stems from the fact that the replacement $x \leftrightarrow z$ generates this set in Table II from the one in Table I, and *vice versa*. The phase-space structure corresponding to the other sets in Table II can be derived in the same manner. However, the interrelation between the structures through qubit permutations is not as easy to unveil in phase space as in physical space, as demonstrated by Fig. 2.

V. THE (3,0,6) MUB-STRUCTURE WIGNER FUNCTION

As we have seen above, individual rotations of the qubits around the x and z axes only generate one MUB structure. To proceed further, one needs to consider simultaneous rotations of two or more qubits. The simplest one, in which we allow simultaneous rotation of the first and the second qubits around the x and z axes, is shown in Table III.

TABLE III: A table defining a (3,0,6) MUB structure.

$\hat{\sigma}_z\hat{\sigma}_z\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_z$	$\hat{\sigma}_z\hat{\mathbb{1}}\hat{\mathbb{1}}$
$\hat{\sigma}_x\hat{\sigma}_x\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_x$	$\hat{\sigma}_x\hat{\mathbb{1}}\hat{\mathbb{1}}$

The rows of the table have the same eigenvectors as the corresponding rows of Tables I and II above. Multiplying the rows columnwise, we obtain the operators $\{\hat{\sigma}_y\hat{\sigma}_y\hat{\mathbb{1}}, \hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_y, \hat{\sigma}_y\hat{\mathbb{1}}\hat{\mathbb{1}}\}$. These operators have the associated phase-space translations $(\mu^3 + \mu^5, \mu^3 + \mu^5)$, (μ^6, μ^6), and (μ^3, μ^3). It is evident that all the three operators make "diagonal" translations in phase space. It is also evident that the corresponding MUB is triseparable: it is a $(3,0,6)$ structure. The striation-generating rays of the phase-state structure are displayed in Fig. 3.

This table has a particularly simple structure although it is built using the same algorithm as all the preceding phase-space structures. The rays in Fig. 3, left side, are really rays, and hence all striations will consist of lines. That is, the different rays are generated by the equations $\beta = \lambda\alpha$ and $\alpha = 0$, where the "slope" λ for each ray is a field element. Moreover, the rays are "visually straight". This, e.g., means that Table III generates the same MUB under the interchange of axes $x \leftrightarrow z$. We can once more generate six "different" tables, representing the same physical structure, by relabeling the qubits. The simplicity of this figure is a chimera, however, because only the rays are diagonal and "visually straight". Translating the rays by, e.g., flipping the first qubit around the z axis (corresponding to a translation by $(0,\mu^3)$ yields the lines in the right part of Fig. 3. In this set of lines, one from each striation, only the hori-

2	6	5	8	3	7	4	9
2	5	4	7	9	6	3	8
2	9	8	4	6	3	7	5
2	7	6	9	4	8	5	3
2	4	3	6	8	5	9	7
2	8	7	3	5	9	6	4
2	3	9	5	7	4	8	6
o	1	1	1	1	1	1	1

2	5	7	6	8	4	9	3
2	7	9	8	3	6	4	5
2	8	3	9	4	7	5	6
2	6	8	7	9	3	1	2
2	9	4	3	5	8	6	7
2	4	6	5	7	3	8	9
2	3	5	4	6	9	7	8
o	1	1	1	1	1	1	1

FIG. 2: The striation-generating lines corresponding to the MUB construction defined by the second and fourth sets in Table 2.

2	8	6	4	9	7	5	3
2	6	4	9	7	5	3	8
2	4	9	7	5	3	8	6
2	9	7	5	3	8	6	4
2	7	5	3	8	6	4	9
2	5	3	8	6	4	9	7
2	3	8	6	4	9	7	5
o	1	1	1	1	1	1	1

2	4	9	7	5	3	8	6
2	7	5	3	8	6	4	9
2	8	6	4	9	7	5	3
o	1	1	1	1	1	1	1
2	6	4	9	7	5	3	8
2	3	8	6	4	9	7	5
2	5	3	8	6	4	9	7
2	9	7	5	3	8	6	4

FIG. 3: The striation-generating rays corresponding to the MUB construction defined by Table 3, left, and the lines resulting from displacement of the rays by $(0,\mu^3)$, right.

TABLE IV: Another table defining a (3,0,6) MUB structure.

$\hat{\sigma}_z\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\sigma}_z\hat{\sigma}_z\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_z$
$\hat{\sigma}_x\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\sigma}_x\hat{\sigma}_x\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_x$

2	8	7	4	5	6	9	3
2	9	8	5	6	7	3	4
2	5	4	8	9	3	6	7
2	6	5	9	3	4	7	8
2	7	6	3	4	5	8	9
2	4	3	7	8	9	5	6
2	3	9	6	7	8	4	5
o	1	1	1	1	1	1	1

FIG. 4: Homogeneous curves corresponding to the MUB construction defined by Table 4.

TABLE V: A table defining a (1,6,2) MUB structure.

$\hat{\mathbb{1}}\hat{\sigma}_z\hat{\mathbb{1}}$	$\hat{\sigma}_z\hat{\sigma}_z\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_z$
$\hat{\sigma}_x\hat{\sigma}_y\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\mathbb{1}}\hat{\sigma}_x$	$\hat{\sigma}_y\hat{\sigma}_z\hat{\mathbb{1}}$

2	7	3	4	8	3	4	9
2	1	7	1	9	4	8	4
2	9	6	6	6	8	7	6
2	5	1	9	5	7	1	8
2	3	8	7	1	1	9	3
2	4	4	8	7	9	5	5
2	8	9	5	3	5	3	7
o	6	5	3	4	6	6	1

FIG. 5: Homogeneous curves corresponding to the MUB construction defined by Table 5.

zontal and the vertical lines remain "visually straight".

The interesting point is that there exist a physically different $(3,0,6)$ MUB structure, also based on two single qubit-rotations and one two-qubit rotation. It is shown in Table IV.

The homogeneous curves corresponding to this structure are displayed in Fig. 4. A significant result is that the structures represented by Tables III and IV are the only physically inequivalent ones that one can obtain from one double rotation and two single qubit rotations.

VI.   MUB STRUCTURES (1,6,2) AND (0,9,0)

It is known that two more MUB structures exist in the three-qubit space, the (1,6,2) and (0,9,0) [34, 35]. The first has only one triseparable basis, whereas the second has none. It is still possible to associate these with Wigner functions in a similar manner, the only difference is that the corresponding Wigner functions will have only one or no "axes" (striations composed of lines). This means that most striations will consist of curves with two, and sometimes four, points on the same horizontal or vertical coordinate. To demonstrate this we use a $(1,6,2)$ structure, which can be generated from the operator sets in Table V.

We see that the set involves "simultaneous" rotations of the first and middle qubits around the x and z axes, effectively resulting in a rotation around the y axis. In phase space this means that by choosing single qubit rotations around the x and z axes as our "primitive" operations, no set of striation-generating translations decouple in horizontal and vertical translations. Therefore, the correspondence with the translation and boost operators $\hat{q}$ and $\hat{p}$, respectively, in the continuous case is lost. The homogeneous curves corresponding to this structure are depicted in Fig. 5. We see that since there is no triseparable basis involving only rotations around the x axis, there is no horizontal ray. Instead, the curve labeled 1

can parametrically be defined as

$$\alpha = \mu^2\kappa^4, \qquad \beta = \mu^2\kappa + \kappa^2 + \mu\kappa^4, \qquad (6.1)$$

which corresponds to the explicit expression

$$\beta^2 + \mu^4\beta = \mu^3\alpha^2 + \mu^2\alpha, \qquad (6.2)$$

or, in an equivalent form,

$$\beta = \mu^3\alpha + \mu^5\alpha^2 + \mu^6\alpha^4. \qquad (6.3)$$

This in turn implies that the other curves have several points on the same horizontal line. For example, curve 6 has four points on the horizontal lines with z-coordinates 0 and μ^4. This reflects the fact that this is a curve invariant under the two horizontal translations $(\mu^3+\mu^6,0)$ and $(\mu^5,0)$. By applying the translation operators $(0,\mu^5)$, $(0,\mu^3+\mu^5)$, $(0,\mu^6)$ one can subsequently generate striations from curves 1 and 3-9. However, this set of translation operators leave ray 2 invariant. To generate the whole striation from ray 2 we can, e.g., use the translation set $(\mu^3+\mu^5,\mu^5)$, $(\mu^6,0)$, and $(\mu^3,\mu^3+\mu^5)$, which are the generating translations of curve 1.

To finally depict a "severely disordered" three-qubit Wigner function (at least to the human eye), we use Table VI, generating a (0,9,0) MUB, which gives rise to the nine striation generating homogeneous curves depicted in Fig. 6. In spite of its disordered appearance, the Wigner function defined from Table VI inherits all the desirable features from the continuous Wigner function except for (visually) straight axes corresponding to q and p. This follows from the fact that all the MUB have entangled basis vectors.

TABLE VI: A table defining a (0,9,0) MUB structure.

$\hat{\sigma}_x\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\sigma}_z\hat{\sigma}_y$	$\hat{\mathbb{1}}\hat{\sigma}_y\hat{\sigma}_z$
$\hat{\sigma}_y\hat{\mathbb{1}}\hat{\mathbb{1}}$	$\hat{\mathbb{1}}\hat{\sigma}_x\hat{\sigma}_z$	$\hat{\mathbb{1}}\hat{\sigma}_z\hat{\sigma}_x$

9	7	3	9	4	7	2	6
4	5	3	6	8	8	5	2
7	6	3	2	5	4	7	5
3	7	8	6	2	7	8	4
6	8	3	8	9	2	9	4
3	6	2	1	4	1	1	1
3	2	5	5	9	4	9	6
o	1	1	9	1	5	7	8

FIG. 6: Homogeneous lines corresponding to the MUB construction defined by Table 6.

VII. CONCLUDING REMARKS

We have discussed the relation between MUBs having different structures with the respect of the entanglement properties and translationally covariant Wigner functions for three qubits. We have shown that there exist three fundamentally different constructions of the discrete Wigner functions if single qubit rotations around two orthogonal axes are used as the generating operations. To construct the Wigner function we need only a MUB generating table and a field-element generating irreducible polynomial. Other constructions can then be generated by qubit permutation and by associating curves in the striations with different states (the choice being arbitrary). The number three is surprisingly small, and shows that requiring translational covariance imposes severe restrictions.

We have also shown that for three-qubit MUBs that have only one or none triseparable bases, the method based on the above-mentioned generating operations still works. Here we do not know how many fundamentally different constructions exist. However, these MUBs are perfectly legitimate from a physical point of view, although they have a more complex appearance than the Wigner functions based on (2,3,4) and (3,0,6) MUBs.

We have not addressed here the interesting question of the factorizability of the Wigner function, that has been previously considered by Durt [50] and Pittenger and Rubin [51]. This is certainly relevant in the framework of quantum tomography, particularly from an experimental viewpoint. In principle, it would suffice that one phase-space point is trifactorizable in order that the other operators are, since the translations themselves are factorizable in virtue of translational covariance. Unfor-

tunately, all our efforts to determine if there exist some triseparable phase-space point operator have failed. To the best of our knowledge, the problem thus still remains open.

In general, the curves that define a MUB on $\mathbb{F}_{2^3}$ form one-dimensional Abelian structures, which can be conveniently parametrized as

$$\alpha(\kappa) = \nu_1\kappa + \nu_2\kappa^2 + \nu_3\kappa^4, \qquad \beta(\kappa) = \eta_1\kappa + \eta_2\kappa^2 + \eta_3\kappa^4, \tag{7.1}$$

where κ is a parameter and the coefficients ν_j and η_j take values on $\mathbb{F}_{2^3}$, so that

$$\alpha(\kappa + \kappa') = \alpha(\kappa) + \alpha(\kappa'),$$
$$\beta(\kappa + \kappa') = \beta(\kappa) + \beta(\kappa'). \tag{7.2}$$

Hence, such curves are the simplest generalization of "straight" Abelian structures (rays) of the form

$$\alpha = 0, \qquad \text{or} \qquad \beta = \lambda\alpha, \tag{7.3}$$

or equivalently

$$\alpha(\kappa) = \eta\kappa, \qquad \beta(\kappa) = \zeta\kappa, \tag{7.4}$$

where η and ζ are fixed field elements.

We finally observe that the structures studied by Wootters [13, 14] (and also by Bandhyopadhyay et al [41] and Durt [40]) always assume two trifactorizable bases, so that only the cases (2,3,4) and (3,0,6) are possible. The existence of a third triseparable basis depends on the choice for the field basis: in the self-dual used in this paper, this is always the case [so we are lead automatically to the (3,0,6) structure], while other choices bring the (2,3,4). In this respect, it is interesting to note that the (3,0,6) is the only three-qubit MUB phase-space structure (depicted in Fig. 3) consisting solely of straight lines.

Acknowledgments

[1] R. J. Glauber, "Coherent and incoherent states of the radiation field," Phys. Rev. **131**, 2766–2788 (1963).
[2] E. C. G. Sudarshan, "Equivalence of semiclassical and quantum mechanical descriptions of statistical light beams," Phys. Rev. Lett. **10**, 277–279 (1963).
[3] K. Husimi, "Some formal properties of the density matrix," Proc. Phys. Math. Soc. Japan **22**, 264–314 (1940).
[4] E. P. Wigner, "On the quantum correction for thermodynamic equilibrium," Phys. Rev. **40**, 749–759 (1932).
[5] A. Vourdas, "Quantum systems with finite Hilbert space," Rep. Prog. Phys. **67**, 267–248 (2004).
[6] J. H. Hannay and M. V. Berry, "Quantization of linear maps on a torus—Fresnel diffraction by a periodic grating," Physica D **1**, 267–290 (1980).

[7] U. Leonhardt, "Quantum-State tomography and discrete Wigner function," Phys. Rev. Lett. **74**, 4101–4105 (1995).

[8] U. Leonhardt, "Discrete Wigner function and quantum-state tomography," Phys. Rev. A **53**, 2998–3013 (1996).

[9] F. A. Buot, "Method for calculating $\mathrm{Tr}\,\mathcal{H}^n$ in solid-state theory," Phys. Rev. B **10**, 3700–3705 (1974).

[10] W. K. Wootters, "A Wigner-function formulation of finite-state quantum mechanics," Ann. Phys. (N.Y.) **176**, 1–21 (1987).

[11] D. Galetti and A. F. R. De Toledo Piza, "An extended Weyl-Wigner transformation for special finite spaces," Physica A **149**, 267–282 (1988).

[12] O. Cohendet, Ph. Combe, M. Sirugue, and M. Sirugue-Collin, "A stochastic treatment of the dynamics of an integer spin," J. Phys. A **21**, 2875–2884 (1988).

[13] W. K. Wootters, "Picturing qubits in phase space," IBM J. Res. Dev. **48**, 99–110 (2004).

[14] K. S. Gibbons, M. J. Hoffman, and W. K. Wootters, "Discrete phase space based on finite fields," Phys. Rev. A **70**, 062101 (2004).

[15] R. Asplund and G. Björk, "Reconstructing the discrete Wigner function and some properties of the measurement bases," Phys. Rev. A **64**, 012106 (2001).

[16] M. Koniorczyk, V. Bužek, and J. Janszky, "Wigner-function description of quantum teleportation in arbitrary dimensions and a continuous limit," Phys. Rev. A **64**, 034301 (2001).

[17] J. P. Paz, "Discrete Wigner functions and the phase-space representation of quantum teleportation," Phys. Rev. A **65**, 062311 (2002).

[18] J. A. Vaccaro and D. T. Pegg, "Wigner function for number and phase," Phys. Rev. A **41**, 5156–5163 (1990).

[19] C. Miquel, J. P. Paz, and M. Saraceno, "Quantum computers in phase space," Phys. Rev. A **65**, 062309 (2002).

[20] Y. Aharonov and B.-G. Englert, "The mean king's problem: spin 1," Z. Naturforsch., A: Phys. Sci. **56a**, 16–19 (2001).

[21] B.-G. Englert and Y. Aharonov, "The mean king's problem: prime degrees of freedom," Phys. Lett. A **284**, 1–5 (2001).

[22] P. K. Aravind, "Solution to the king's problem in prime power dimensions," Z. Naturforsch., A: Phys. Sci. **58a**, 85–92 (2003).

[23] P. K. Aravind, "Best conventional solutions to the king's problem," Z. Naturforsch., A: Phys. Sci. **58a**, 682–690 (2003).

[24] T. Durt, "Bell states, mutually unbiased bases and the mean king's problem," e-print quant-ph/0401037.

[25] A. Hayashi, M. Horibe, and T. Hashimoto, "Mean king's problem with mutually unbiased bases and orthogonal Latin squares," Phys. Rev. A **71**, 052331 (2005).

[26] P. Delsarte, J. M. Goethals, and J. J. Seidel, "Bounds for systems of lines and Jacobi polynomials," Philips Res. Rep. **30**, 91–105 (1975).

[27] W. K. Wootters, "Quantum mechanics without probability amplitudes," Found. Phys. **16**, 391–405 (1986).

[28] W. K. Wootters and B. D. Fields, "Optimal state-determination by mutually unbiased measurements," Ann. Phys. (N.Y.) **191**, 363–381 (1989).

[29] C. H. Bennett and G. Brassard, in *Proceedings of the IEEE International Conference on Computers, Systems, and Signal Processing, Bangalore, India* (IEEE, New York, 1984) pp. 175–179.

[30] H. Bechmann-Pasquinucci and A. Peres, "Quantum cryptography with 3-state systems," Phys. Rev. Lett. **85**, 3313–3316 (2000).

[31] R. Asplund, G. Björk, and M. Bourennane, "An expectation value expansion of Hermitian operators in a discrete Hilbert space," J. Opt. B: Quantum Semiclass. Opt. **3**, 163–170 (2001).

[32] D. Bruß and C. Macchiavello, "Optimal eavesdropping in cryptography with three-dimensional quantum states," Phys. Rev. Lett. **88**, 127901 (2002).

[33] I. D. Ivanović, "Geometrical description of quantal state determination," J. Phys. A **14**, 3241–3246 (1981).

[34] J. Lawrence, Č. Brukner, and A. Zeilinger, "Mutually unbiased binary observable sets on N qubits," Phys. Rev. A **65**, 032320 (2002).

[35] J. L. Romero, G. Björk, A. B. Klimov, and L. L. Sánchez-Soto, "Structure of the sets of mutually unbiased bases for N qubits," Phys. Rev. A **72**, 062310 (2005).

[36] A. R. Calderbank, E. M. Rains, P. W. Shor, and N. J. A. Sloane, "Quantum error correction and orthogonal geometry," Phys. Rev. Lett. **78**, 405–408 (1997).

[37] A. R. Calderbank, P. J. Cameron, W. M. Cantor, and J. J. Seidel, "$\mathbb{Z}_4$-Kerdock codes, orthogonal spreads, and extremal Euclidean line-sets," Proc. London Math. Soc. **75**, 436–480 (1997).

[38] S. Chaturvedi, "Aspects of mutually unbiased bases in odd-prime-power dimensions," Phys. Rev. A **65**, 044301 (2002).

[39] A. O. Pittenger and M. H. Rubin, "Mutually unbiased bases, generalized spin matrices and separability," Linear Algebr. Appl. **390**, 255–278 (2004).

[40] T. Durt, "About mutually unbiased bases in even and odd prime power dimensions," J. Phys. A **38**, 5267–5284 (2005).

[41] S. Bandyopadhyay, P. O. Boykin, V. Roychowdhury, and F. Vatan, "A new proof for the existence of mutually unbiased bases," Algorithmica **34**, 512–528 (2002).

[42] A. B. Klimov, L. L. Sánchez-Soto, and H. de Guise, "Multicomplementary operators via finite Fourier transform," J. Phys. A **38**, 2747–2760 (2005).

[43] M. Grassl, "On SIC-POVMs and MUBs in dimension 6," e-print quant-ph/0406175.

[44] P. Wocjan and T. Beth, "New construction of mutually unbiased bases in square dimensions," Quantum Inform. Compu. **5**, 93–101 (2005).

[45] C. Archer, "There is no generalization of known formulas for mutually unbiased bases," J. Math. Phys. **46**, 022106 (2005).

[46] A. Klappenecker and M. Rötteler, "Constructions of mutually unbiased bases," Lect. Notes Comput. Sc. **2948**, 137–144 (2004).

[47] W. K. Wootters, "Quantum measurements and finite geometry," Found. Phys. **36**, 112–126 (2006).

[48] I. Bengtsson and A. Ericsson, "Mutually unbiased bases and the complementarity polytope," Open Syst. Inf. Dyn. **12**, 107–120 (2005).

[49] M. Saniga, M. Planat, and H. Rosu, "Mutually unbiased bases and finite projective planes," J. Opt. B: Quantum Semiclass. Opt. **6** L19–L20 (2004).

[50] T. Durt, "Tomography of one and two qubit states and factorisation of the Wigner distribution in prime power dimensions," e-print quant-ph/0604117.

[51] A. O. Pittenger and M. H. Rubin, "Wigner function and separability for finite systems," J. Phys. A **38**, 6005–6036

(2005).